

BPS Black Holes

Bernard de Wit^a

^aInstitute for Theoretical Physics & Spinoza Institute,
Utrecht University, Utrecht, The Netherlands

The entropy of BPS black holes in four space-time dimensions is discussed from both macroscopic and microscopic points of view.

1. INTRODUCTION

Classical black holes are solutions of Einstein's equations of general relativity that exhibit an event horizon. From inside this horizon, nothing (and in particular, no light) can escape. The region inside the horizon is therefore not in the backward lightcone of future timelike infinity. However, since the discovery of Hawking radiation [1], it has become clear that many of the classical features of black holes will be subject to change.

In these lecture we consider static, spherically symmetric black holes in four space-time dimensions that carry electric and/or magnetic charges with a flat space-time geometry at spatial infinity. Such solutions exist in Einstein-Maxwell theory, the classical field theory of gravity and electromagnetism. The most general static black holes of this type correspond to the Reissner-Nordstrom solutions and are characterized by a charge Q and a mass M . In the presence of magnetic charges, Q is replaced by $\sqrt{q^2 + p^2}$ in most formulae, where q and p denote the electric and the magnetic charge, respectively. Hence there is no need to distinguish between the two types of charges. For zero charges one is dealing with Schwarzschild black holes.

Two quantities associated with the black hole horizon are its area A and the surface gravity κ_s . The area is simply the area of the two-sphere defined by the horizon. The surface gravity, which is constant on the horizon, is related to the force (measured at spatial infinity) that holds a unit test mass in place. On the other hand, the mass

M and charge Q of the black hole are not primarily associated with the horizon and are defined in terms of appropriate surface integrals at spatial infinity.

As is well known, there exists a striking correspondence between the laws of thermodynamics and the laws of black hole mechanics [2]. Of particular importance is the first law, which, for thermodynamics, states that the variation of the total energy is equal to the temperature times the variation of the entropy, modulo work terms, for instance proportional to a change of the volume. The corresponding formula for black holes expresses how the variation of the black hole mass is related to the variation of the horizon area, up to work terms proportional to the variation of the angular momentum. In addition there can also be a term proportional to a variation of the charge, multiplied by the electric/magnetic potential μ at the horizon. Specifically, the first law of thermodynamics, $\delta E = T \delta S - p \delta V$, translates into

$$\delta M = \frac{\kappa_s}{2\pi} \frac{\delta A}{4} + \mu \delta Q + \Omega \delta J. \quad (1)$$

The reason for factorizing the first term on the right-hand side in this particular form, is that $\kappa_s/2\pi$ equals the Hawking temperature [1]. This then leads to the identification of the black hole entropy in terms of the horizon area,

$$S_{\text{macro}} = \frac{1}{4} A, \quad (2)$$

a result that is known as the area law [3]. In these equations the various quantities have been defined in Planck units, meaning that they have

been made dimensionless by multiplication with an appropriate power of Newton's constant (we will set $\hbar = c = 1$). This constant appears in the Einstein-Hilbert Lagrangian according to $\mathcal{L}_{\text{EH}} = -(16\pi G_{\text{N}})^{-1} \sqrt{|g|} R$. With this normalization the quantities appearing in the first law are independent of the scale of the metric.

Einstein-Maxwell theory can be naturally embedded into $N = 2$ supergravity. This supergravity theory has possible extensions with several abelian gauge fields and a related number of massless scalar fields (often called 'moduli' fields, for reasons that will become clear later on). At spatial infinity these moduli fields will tend to a constant, and the black hole mass will depend on these constants, thus introducing additional terms on the right-hand side of (1).

For Schwarzschild black holes the only relevant parameter is the mass M and we note the following relations,

$$A = 16\pi M^2, \quad \kappa_s = \frac{1}{4M}, \quad (3)$$

consistent with (1). For the Reissner-Nordstrom black hole, the situation is more subtle. Here one distinguishes three different cases. For $M > Q$ one has non-extremal solutions, which exhibit two horizons, an exterior event horizon and an interior horizon. When $M = Q$ one is dealing with an extremal black hole, for which the two horizons coalesce and the surface gravity vanishes. In that case one has

$$A = 4\pi M^2, \quad \kappa_s = 0, \quad \mu = Q \sqrt{\frac{4\pi}{A}}. \quad (4)$$

It is straightforward to verify that this result is consistent with (1) for variations $\delta M = \delta Q$ and $\kappa_s = 0$. Because the surface gravity vanishes, one might expect the entropy to vanish as well, as is suggested by the third law of thermodynamics. However, this is not the case because the horizon area remains finite for zero surface gravity. Finally, solutions with $M < Q$ are not regarded as physically acceptable. Their total energy is less than the electromagnetic energy alone and they no longer have an event horizon but exhibit a naked singularity. Hence extremal black holes

saturate the bound $M \geq Q$ for physically acceptable black hole solutions.

When embedding Einstein-Maxwell theory into a complete supergravity theory, the above classification has an interpretation in terms of the supersymmetry algebra. This algebra has a central extension proportional to the black hole charge(s). Unitary representations of the supersymmetry algebra must necessarily have masses that are larger than or equal to the charge. When this bound is saturated, one is dealing with so-called BPS supermultiplets. Such supermultiplets are smaller than the generic massive $N = 2$ supermultiplets and have a different spin content. Because of this, BPS states are stable under (adiabatic) changes of the coupling constants and the relation between charge and mass remains preserved. This important feature of BPS states will be relevant for what follows. In these lectures BPS black hole solutions are defined by the fact that they have some residual supersymmetry, so that they saturate a bound implied by the supersymmetry algebra.

So far we did not refer to the explicit Schwarzschild or Reissner-Nordstrom black hole solutions, which can be found in many places in the literature. One feature that should be stressed, concerns the near-horizon geometry. For extremal, static and spherically symmetric black holes, this geometry is restricted to the product of the sphere S^2 and an anti-de Sitter space AdS_2 , corresponding to the line element,

$$ds^2 = -r^2 dt^2 + \frac{dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)}{r^2}. \quad (5)$$

In these coordinates the horizon is located at $r = 0$, where the timelike Killing vector $K = \partial_t$ turns lightlike. Such a horizon is called a Killing horizon.

In these notes we discuss various aspects of the relation between black hole solutions and corresponding microscopic descriptions. Section 2 generally describes the macroscopic (field theoretic) and microscopic (statistical) approach to black hole entropy and indicates why they are related. Section 3 summarizes the calculation of the black hole entropy based on a fivebrane wrapping on a Calabi-Yau four-cycle in a compactification of

M-Theory on the product space of a Calabi-Yau threefold and a circle. In section 4 the attractor equations are discussed for extremal black holes on the basis of a variational principle defined for a generic gravitational theory. Section 5 contains a brief review of $N = 2$ supergravity and the superconformal multiplet calculus. This material is used in the description of the BPS attractor equations in $N = 2$ supergravity presented in section 6. In this case there also exists a corresponding formulation in terms of a BPS entropy function. Finally section 7 discusses the relation between this entropy function and the black hole partition function.

We try to concentrate as much as possible on the conceptual basis of the underlying ideas, but these notes cannot do justice to all aspects relevant for the study of black holes in string theory, such as use of the $\text{AdS}_3/\text{CFT}_2$ correspondence, black holes or related objects in other than four space-time dimensions, the relation with indices for BPS states, and the like. We refer to some recent reviews where some of these topics have been discussed [4,5,6,7].

2. DUAL PERSPECTIVES

A central question in black hole thermodynamics concerns the statistical interpretation of the black hole entropy. String theory has provided new insights here [8], which enable the identification of the black hole entropy as the logarithm of the degeneracy of states $d(Q)$ of charge Q belonging to a certain system of microstates. In string theory these microstates are provided by the states of wrapped brane configurations of given momentum and winding. When calculating the black hole solutions in the corresponding effective field theory with the charges specified by the brane configuration, one discovers that the black hole area is equal to the logarithm of the brane state degeneracy, at least in the limit of large charges. We will be reviewing some aspects of this remarkable correspondence here.

The horizon area, which is expected to be proportional to the macroscopic entropy according to the Bekenstein-Hawking area law, turns out to grow quadratically with the charges Q . After

converting to string units the radius of a black hole is therefore proportional to

$$\frac{R_{\text{horizon}}}{l_{\text{string}}} \sim g_s Q, \quad (6)$$

where l_{string} and g_s are the string length and coupling constant, respectively. Since we will be assuming that the charges are large, the black holes are generically much larger than the string scale. Consequently these black holes are called *large* and can be identified with the macroscopic black holes we have been discussing earlier. However, there are also situations where the leading contribution to the area is only linear in the charges Q . In that case (6) is replaced by

$$\frac{R_{\text{horizon}}}{l_{\text{string}}} \sim g_s \sqrt{Q}, \quad (7)$$

Moreover, in that case the string coupling (inversely proportional to the dilaton field) cannot be taken constant, but tends to zero for large charges according to $g_s \sim Q^{-1/2}$, so that the radius of the black hole remains comparable to the string scale. These black holes are called *small*. Their corresponding classical supergravity solutions exhibit a vanishing horizon area and a dilaton field that diverges at the horizon. To reliably compute the right-hand side of (7) therefore requires to include appropriate terms in the effective action of higher order in space-time derivatives. We return to this issue in subsection 6.2.

To further understand the relation between a field-theoretic description and a microscopic description it is relevant that strings live in more than four space-time dimensions. In most situations the extra dimensions are compactified on some internal manifold X and one is dealing with the standard Kaluza-Klein scenario leading to effective field theories in four dimensions, describing low-mass modes of the fields associated with appropriate eigenfunctions on the internal manifold. Locally, the original space-time is a product $M^4 \times X$, where M^4 denotes the four-dimensional space-time that we experience in daily life. In this situation there exists a corresponding space X at every point x^μ of M^4 , whose size is such that it will not be directly observable. However, this

space X does not have to be the same at every point in M^4 , and moving through M^4 one may encounter various spaces X that are not necessarily equivalent. In principle they belong to some well-defined class of fixed topology parametrized by certain moduli. These moduli will appear as fields in the four-dimensional effective field theory. For instance, suppose that the spaces X are n -dimensional tori T^n . The metric of T^n will appear as a field in the four-dimensional theory and is related to the torus moduli. Hence, when dealing with a solution of the four-dimensional theory that is not constant in M^4 , each patch in M^4 has a non-trivial image in the space of moduli that parametrize the internal spaces X .

Let us now return to a black hole solution, viewed in this higher-dimensional perspective. The fields, and in particular the four-dimensional space-time metric, will vary nontrivially over M^4 , and so will the internal spaces X . When moving to the center of the black hole the gravitational fields will become strong and the local product structure into $M^4 \times X$ could break down. Conventional Kaluza-Klein theory does not have much to say about what happens, beyond the fact that the four-dimensional solution can be lifted to the higher-dimensional one, at least in principle.

However, there is a feature of string theory that is absent in a purely field-theoretic approach. In the effective field-theoretic context only the local degrees of freedom of strings and branes are captured. But extended objects may also carry global degrees of freedom, as they can also wrap themselves around non-trivial cycles of the internal space X . This wrapping tends to take place at a particular position in M^4 , so in the context of the four-dimensional effective field theory this will reflect itself as a pointlike object. The wrapped object is the string theory representation of the black hole!

We are thus dealing with two complementary pictures of the black hole. One is based on general relativity where a point mass generates a global solution of space-time with strongly varying gravitational fields, which we shall refer to as the *macroscopic* description. The other one, based on the internal space where an extended object is entangled in one of its cycles, does not immedi-

ately involve gravitational fields and can easily be described in flat space-time. This description will be referred to as *microscopic*. To understand how these two descriptions are related is far from easy, but a connection must exist in view of the fact that gravitons are closed string states which interact with the wrapped branes. These interactions are governed by the string coupling constant g_s and we are thus confronted with an interpolation in that coupling constant. In principle, such an interpolation is very difficult to carry out, so that a realistic comparison between microscopic and macroscopic results is usually impossible. However, reliable predictions are possible for extremal black holes that are BPS! As we indicated earlier, in that situation there are reasons to trust such interpolations. Indeed, it has been shown that the predictions based on these two alternative descriptions can be successfully compared and new insights about black holes can be obtained.

But how do the wrapped strings and branes represent themselves in the effective action description and what governs their interactions with the low-mass fields? Here it is important to realize that the massless four-dimensional fields are associated with harmonic forms on X . Harmonic forms are in one-to-one correspondence with so-called cohomology groups consisting of equivalence classes of forms that are closed but not exact. The number of independent harmonic forms of a given degree is given by the so-called Betti numbers, which are fixed by the topology of the spaces X . When expanding fields in a Kaluza-Klein scenario, the number of corresponding massless fields can be deduced from an expansion in terms of tensors on X corresponding to the various harmonic forms. The higher-dimensional fields $\Phi(x, y)$ thus decompose into the massless fields $\phi^A(x)$ according to (schematically),

$$\Phi(x, y) = \phi^A(x) \omega_A(y), \quad (8)$$

where $\omega_A(y)$ denotes the independent harmonic forms on X . The above expression, when substituted into the action of the higher-dimensional theory, leads to interactions of the fields ϕ^A proportional to the ‘coupling constants’,

$$C_{ABC\dots} \propto \int_X \omega_A \wedge \omega_B \wedge \omega_C \dots \quad (9)$$

These constants are known as intersection numbers, for reasons that will become clear shortly.

We already mentioned that the Betti numbers depend on the topology of X . This is related to Poincaré duality, according to which cohomology classes are related to homology classes. The latter consist of submanifolds of X without boundary that are themselves not a boundary of some other submanifold of X . This is precisely relevant for wrapped branes which indeed cover submanifolds of X , but are not themselves the boundary of a submanifold because otherwise the brane could collapse to a point. Without going into detail, this implies that there exists a dual relationship between harmonic p -forms ω and $(d_X - p)$ -cycles, where d_X denotes the dimension of X . We can therefore choose a homology basis for the $(d_X - p)$ -cycles dual to the basis adopted for the p -forms. Denoting this basis by Ω_A the wrapping of an extended object can now be characterized by specifying its corresponding cycle \mathcal{P} in terms of the homology basis,

$$\mathcal{P} = p^A \Omega_A. \quad (10)$$

The integers p^A count how many times the extended object is wrapped around the corresponding cycle, so we are actually dealing with integer-valued cohomology and homology. The wrapping numbers p^A reflect themselves as magnetic charges in the effective action. The electric charges are already an integer part of the effective action, because they are associated with gauge transformations that usually originate from the higher-dimensional theory.

Owing to Poincaré duality it is thus very natural that the winding numbers interact with the massless modes in the form of magnetic charges, so that they can be incorporated in the effective action. Before closing this section, we note that, by Poincaré duality, we can express the number of intersections by

$$P \cdot P \cdot P \cdots = C_{ABC\cdots} p^A p^B p^C \cdots. \quad (11)$$

This is a topological characterization of the wrapping, which will appear in later formulae.

3. BLACK HOLES IN M/STRING THEORY – AN EXAMPLE

As an example we now discuss the black hole entropy derived from microscopic arguments in a special case. In a later section we will consider the corresponding expression for the macroscopic entropy. We start from M-theory, which is the strong coupling limit of type-IIA string theory. Its massless states are described by eleven-dimensional supergravity. The latter is invariant under 32 supersymmetries. Seven of the eleven space-time dimensions are compactified on an internal space which is the product of a Calabi-Yau threefold (a Ricci flat three-dimensional complex manifold, henceforth denoted by CY_3) times a circle S^1 . Such a space induces a partial breaking of supersymmetry which leaves 8 supersymmetries unaffected. In the context of the four-dimensional space-time M^4 , these 8 supersymmetries are encoded into two independent Lorentz spinors and for that reason this symmetry is referred to as $N = 2$ supersymmetry. Hence the effective four-dimensional field theory will be some version of $N = 2$ supergravity.

M-theory contains a five-brane and this is the microscopic object that is responsible for the black holes that we consider: the five-brane has wrapped itself on a 4-cycle \mathcal{P} of the CY_3 space [9]. Alternatively one may study this class of black holes in type-IIA string theory, with a 4-brane wrapping the 4-cycle [10]. The 4-cycle is subject to certain requirements which will be mentioned in due course.

The massless modes captured by the effective field theory correspond to harmonic forms on the CY_3 space; they do not depend on the S^1 coordinate. The 2-forms are of particular interest. In the effective theory they give rise to vector gauge fields A_μ^A , which originate from the rank-three tensor gauge field in eleven dimensions. In addition there is an extra vector field A_μ^0 corresponding to a 0-form which is related to the graviphoton associated with S^1 . This field will couple to the electric charge q_0 associated with momentum modes on S^1 in the standard Kaluza-Klein fashion. The 2-forms are dual to 4-cycles and the wrapping of the five-brane is encoded in

terms of the wrapping numbers p^A , which appear in the effective field theory as magnetic charges coupling to the gauge fields A_μ^A . Here we see Poincaré duality at work, as the magnetic charges couple nicely to the corresponding gauge fields. In view of the fact that the product of three 2-forms defines a 6-form that can be integrated over the CY_3 space, there exist non-trivial triple intersection numbers C_{ABC} . These numbers will appear in the three-point couplings of the effective field theory. There is a subtle topological feature that we have not explained before, which is typical for complex manifolds containing 4-cycles, namely the existence of another quantity of topological interest known as the second Chern class. The second Chern class is a 4-form whose integral over a four-dimensional Euclidean space defines the instanton number. The 4-form can be integrated over the 4-cycle \mathcal{P} and yields $c_{2A} p^A$, where the c_{2A} are integers.

Let us now turn to the microscopic counting of degrees of freedom [9]. These degrees of freedom are associated with the massless excitations of the wrapped five-brane characterized by the wrapping numbers p^A on the 4-cycle. The 4-cycle \mathcal{P} must correspond to a holomorphically embedded complex submanifold in order to preserve 4 supersymmetries. The massless excitations of the five-brane are then described by a $(1+1)$ -dimensional superconformal field theory (the reader may also consult [11]). Because we have compactified the spatial dimension on S^1 , we are dealing with a closed string with left- and right-moving states. The 4 supersymmetries of the conformal field theory reside in one of these two sectors, say the right-handed one. Conformal theories in $1+1$ dimensions are characterized by a central charge, and in this case there is a central charge for the right- and for the left-moving sector separately. The two central charges are expressible in terms of the wrapping numbers p^A and depend on the intersection numbers and the second Chern class, according to

$$\begin{aligned} c_L &= C_{ABC} p^A p^B p^C + c_{2A} p^A, \\ c_R &= C_{ABC} p^A p^B p^C + \frac{1}{2} c_{2A} p^A. \end{aligned} \quad (12)$$

Here we should stress that the above result is far from obvious and holds only under the condition

that the p^A are large. In that case every generic deformation of \mathcal{P} will be smooth. Under these circumstances it is possible to relate the topological properties of the 4-cycle to the topological data of the Calabi-Yau space.

We can now choose a state of given momentum q_0 which is supersymmetric in the right-moving sector. From rather general arguments it follows that such states exist. The corresponding states in the left-moving sector have no bearing on the supersymmetry and these states have a certain degeneracy depending on the value of q_0 . In this way we have a tower of degenerate BPS states invariant under 4 supersymmetries, built on a supersymmetric state in the right-moving sector. We can then use Cardy's formula, which states that the degeneracy of states for fixed but large momentum (large as compared to c_L) equals $\exp[2\pi\sqrt{|q_0| c_L/6}]$. This leads to the following expression for the entropy,

$$\mathcal{S}_{\text{micro}}(p, q) = 2\pi\sqrt{\frac{1}{6}|\hat{q}_0|(C_{ABC} p^A p^B p^C + c_{2A} p^A)}, \quad (13)$$

where q_0 has been shifted according to

$$\hat{q}_0 = q_0 + \frac{1}{2} C^{AB} q_A q_B. \quad (14)$$

Here C^{AB} is the inverse of $C_{AB} = C_{ABC} p^C$. This modification is related to the fact that the electric charges associated with the gauge fields A_μ^A will interact with the M-theory two-brane [9]. The existence of this interaction can be inferred from the fact that the two-brane interacts with the rank-three tensor field in eleven dimensions, from which the vector gauge fields A_μ^A originate.

We stress that the above results apply in the limit of large charges. The first term proportional to the triple intersection number is obviously the leading contribution whereas the terms proportional to the second Chern class are subleading. The importance of the subleading terms will become more clear in later sections. Having obtained a microscopic representation of a BPS black hole, it now remains to make contact with it by deriving the corresponding black hole solution directly in the $N = 2$ supergravity theory. This is discussed in some detail in section 6.

4. ATTRACTOR EQUATIONS

The microscopic expression for the black hole entropy depends only on the charges, whereas a field-theoretic calculation can in principle depend on other quantities, such as the values of the moduli fields at the horizon. To establish any agreement between these two approaches, the moduli (as well as any other relevant fields that enter the calculation) must take fixed values at the horizon which may depend on the charges. As it turns out this is indeed the case for extremal black hole solutions, as was first demonstrated in the context of $N = 2$ supersymmetric black holes [12,13,14]. The values taken by the fields at the horizon are independent of their asymptotic values at spatial infinity. This fixed point behaviour is encoded in so-called attractor equations.

In the presence of higher-derivative interactions it is very difficult to explicitly construct black hole solutions and to exhibit the attractor phenomenon. However, by concentrating on the near-horizon region one can usually determine the fixed-point values directly without considering the interpolation between the horizon and spatial infinity. Provided the symmetry of the near-horizon region is restrictive enough, the attractor phenomenon can be described conveniently in terms of a variational principle for a so-called entropy function. The stationarity of this entropy function then yields the attractor equations and its value at the attractor point equals the macroscopic entropy. The purpose of the present section is to explain this phenomenon.

For $N = 2$ BPS black holes with higher-derivative interactions the attractor equations follow from classifying possible solutions with full supersymmetry [15]. As it turns out supersymmetry determines the near-horizon geometry (and thus the horizon area), the values of the moduli fields in terms of the charges and the value of the entropy as defined by the Noether charge definition of Wald [16]. This result can also be described in terms of a variational principle [17,18], as we shall explain in section 6.

Let us now turn to the more generic case of extremal, static, spherically symmetric black holes that are not necessarily BPS, following the ap-

proach of [19] which is based on an action principle. Similar approaches can be found elsewhere in the literature, for instance, in [20,21,22]. When dealing with spherically symmetric solutions, one integrates out the spherical degrees of freedom and obtains a reduced action for a $1 + 1$ dimensional field theory which fully describes the black hole solutions. Here we consider a general system of abelian vector gauge fields, scalar and other matter fields coupled to gravity. The geometry is thus restricted to the product of the sphere S^2 and a $1 + 1$ dimensional space-time, and the dependence of the fields on the S^2 coordinates θ and φ is fixed by symmetry arguments. For the moment we will not make any assumption regarding the dependence on the remaining two coordinates r and t . Consequently we write the general field configuration consistent with the various isometries as

$$\begin{aligned} ds^2_{(4)} &= g_{\mu\nu} dx^\mu dx^\nu \\ &= ds^2_{(2)} + v_2 \left(d\theta^2 + \sin^2 \theta d\varphi^2 \right), \\ F_{rt}{}^I &= e^I, \quad F_{\theta\varphi}{}^I = \frac{p^I}{4\pi} \sin \theta. \end{aligned} \quad (15)$$

The $F_{\mu\nu}{}^I$ denote the field strengths associated with a number of abelian gauge fields. The θ -dependence of $F_{\theta\varphi}{}^I$ is fixed by rotational invariance and the p^I denote the magnetic charges. The latter are constant by virtue of the Bianchi identity, but all other fields are still functions of r and t . As we shall see in a moment the fields e^I are dual to the electric charges. The radius of S^2 is defined by the field v_2 . The line element of the $1 + 1$ dimensional space-time will be expressed in terms of the two-dimensional metric \bar{g}_{ij} , whose determinant will be related to a field v_1 according to,

$$v_1 = \sqrt{|\bar{g}|}. \quad (16)$$

Eventually \bar{g}_{ij} will be taken proportional to an AdS_2 metric,

$$ds^2_{(2)} = \bar{g}_{ij} dx^i dx^j = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right). \quad (17)$$

In addition to the fields e^I , v_1 and v_2 , there may be a number of other fields which for the moment we denote collectively by u_α .

As is well known theories based on abelian vector fields are subject to electric/magnetic duality, because their equations of motion expressed in terms of the dual field strengths,¹

$$G_{\mu\nu I} = \sqrt{|g|} \varepsilon_{\mu\nu\rho\sigma} \frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}^I}, \quad (18)$$

take the same form as the Bianchi identities for the field strengths $F_{\mu\nu}^I$. Adopting the conventions where $x^\mu = (t, r, \theta, \varphi)$ and $\varepsilon_{tr\theta\varphi} = 1$, and the signature of the space-time metric equals $(-, +, +, +)$, which is consistent with (17), it follows that, in the background (15),

$$\begin{aligned} G_{\theta\varphi I} &= -v_1 v_2 \sin \theta \frac{\partial \mathcal{L}}{\partial e^I}, \\ G_{rt I} &= -4\pi v_1 v_2 \frac{\partial \mathcal{L}}{\partial p^I}. \end{aligned} \quad (19)$$

These two tensors can be written as $q_I \sin \theta / (4\pi)$ and f_I . The quantities q_I and f_I are conjugate to p^I and e^I , respectively, and can be written as

$$\begin{aligned} q_I(e, p, v, u) &= -4\pi v_1 v_2 \frac{\partial \mathcal{L}}{\partial e^I}, \\ f_I(e, p, v, u) &= -4\pi v_1 v_2 \frac{\partial \mathcal{L}}{\partial p^I}. \end{aligned} \quad (20)$$

They depend on the constants p^I and on the fields e^I , $v_{1,2}$ and u_α , and possibly their t and r derivatives, but no longer on the S^2 coordinates θ and φ . Upon imposing the field equations it follows that the q_I are constant and correspond to the electric charges. Our aim is to obtain a description in terms of the charges p^I and q_I , rather than in terms of the p^I and e^I .

Electric/magnetic duality transformation are induced by rotating the tensors $F_{\mu\nu}^I$ and $G_{\mu\nu I}$ by a constant transformation, so that the new linear combinations are all subject to Bianchi identities. Half of them are then selected as the new field strengths defined in terms of new gauge fields, while the Bianchi identities on the remaining linear combinations are regarded as field equations belonging to a new Lagrangian defined in terms

of the new field strengths. In order that this dualization can be effected the rotation of the tensors must belong to $\text{Sp}(2n+2; \mathbb{R})$, where $n+1$ denotes the number of independent gauge fields [23]. Naturally the duality leads to new quantities $(\tilde{p}^I, \tilde{q}_I)$ and $(\tilde{e}^I, \tilde{f}_I)$, related to the original ones by the same $\text{Sp}(2n+2; \mathbb{R})$ rotation. Since the charges are not continuous but will take values in an integer-valued lattice, this group should eventually be restricted to an appropriate arithmetic subgroup.

Subsequently we define the reduced Lagrangian by the integral of the full Lagrangian over S^2 ,

$$\mathcal{F}(e, p, v, u) = \int d\theta d\varphi \sqrt{|g|} \mathcal{L}, \quad (21)$$

We note that the definition of the conjugate quantities q_I and f_I takes the form,

$$q_I = -\frac{\partial \mathcal{F}}{\partial e^I}, \quad f_I = -\frac{\partial \mathcal{F}}{\partial p^I}. \quad (22)$$

It is known that a Lagrangian does not transform as a function under electric/magnetic dualities (see, e.g. [24]), but one can generally show that the following combination does [25],

$$\mathcal{E}(q, p, v, u) = -\mathcal{F}(e, p, v, u) - e^I q_I. \quad (23)$$

More precisely, this quantity transforms under electric/magnetic duality according to $\tilde{\mathcal{E}}(\tilde{q}, \tilde{p}, v, u) = \mathcal{E}(q, p, v, u)$. In view of the first equation (22), the definition (23) takes the form of a Legendre transform. Furthermore the field equations imply that the q_I are constant and that the action, $\int dt dr \mathcal{E}$, is stationary under variations of the fields v and u , while keeping the p^I and q_I fixed. This is to be expected as \mathcal{E} is in fact minus the Hamiltonian density associated with the reduced Lagrangian density (21), at least as far as the vector fields are concerned.

In the near-horizon background (17), assuming fields that are invariant under the AdS_2 isometries, the generally covariant derivatives of the fields vanish and the function \mathcal{E} depends only on the fields which no longer depend on the coordinates. The equations of motion then imply that the values of the fields $v_{1,2}$ and u_α are determined

¹We assume that the Lagrangian is a function of the abelian field strengths and does not depend explicitly on the gauge fields.

by demanding \mathcal{E} to be stationary under variations of v and u ,

$$\frac{\partial \mathcal{E}}{\partial v} = \frac{\partial \mathcal{E}}{\partial u} = 0, \quad q_I = \text{constant}. \quad (24)$$

The function $2\pi \mathcal{E}(q, p, v, u)$ then coincides with the entropy function proposed by Sen [19]. The first two equations of (24) are interpreted as the attractor equations. At the attractor point one may prove

$$\mathcal{E}|_{\text{attractor}} = - \int d\theta d\varphi \sqrt{|g|} R_{rtrt} \frac{\partial \mathcal{L}}{\partial R_{rtrt}}, \quad (25)$$

where the right-hand side is evaluated in the near-horizon geometry. This leads to the expression

$$2\pi \mathcal{E}|_{\text{attractor}} = 2\pi \int_{\Sigma_{\text{hor}}} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \varepsilon_{\mu\nu} \varepsilon_{\rho\sigma}, \quad (26)$$

where Σ_{hor} denotes a spacelike cross section of the Killing horizon, and $\varepsilon_{\mu\nu}$ the normal bivector which acts in the space normal to Σ_{hor} . This is precisely the expression for the Wald entropy [16] (applied to this particular case). For the Einstein-Hilbert action, (26) equals a quarter of the horizon area in Planck units. For more general Lagrangians (26) may lead to deviations from the area law, as we will see in due course. Note that the entropy function does not necessarily depend on all fields at the horizon. The values of some of the fields will then be left unconstrained, but those will not appear in the expression for the Wald entropy either.

The above derivation of the entropy function applies to any gauge and general coordinate invariant Lagrangian, including Lagrangians with higher-derivative interactions. In the absence of higher-derivative terms, the reduced Lagrangian \mathcal{F} is at most quadratic in e^I and p^I and the Legendre transform (23) can easily be carried out. The results coincide with corresponding terms in the so-called black hole potential discussed in e.g. [20,21,22].

5. N=2 SUPERGRAVITY

In the previous section the symmetry of the near-horizon geometry played a crucial role. For BPS black holes the supersymmetry enhancement

at the horizon is the crucial input that constrains both some of the fields at the horizon as well as the near-horizon geometry itself. The black hole solutions that we will be considering have a residual $N = 1$ supersymmetry (so that they are BPS) and are solitonic interpolations between $N = 2$ configurations at the horizon and at spatial infinity. Obviously to describe BPS black holes one needs to consider extended supergravity theories. In view of the application described in section 3, $N = 2$ supergravity is relevant. Moreover, $N = 2$ supergravity has off-shell formulations (meaning that supersymmetry transformations realize the supersymmetry algebra without the need for imposing field equations associated with a specific Lagrangian) and this facilitates the calculations in an essential way. This aspect is especially important because we will be considering supersymmetric Lagrangians with interactions containing more than two derivatives.

In the following subsections we present a brief introduction to $N = 2$ supergravity. Supermultiplets are introduced in subsection 5.1 and supersymmetric actions in subsection 5.2. Finally, in subsection 5.3, we elucidate the use of compensating fields and corresponding supermultiplets to familiarize the reader with the principles underlying the superconformal multiplet calculus. Further details can be found in the literature [26,27].

5.1. Supermultiplets

In this subsection we briefly introduce the supermultiplets that play a role in the following. Of particular interest are the vector and the Weyl supermultiplet. Other multiplets are the tensor supermultiplets and the hypermultiplets, but those will not be discussed as they only play an ancillary role.

The covariant fields and field strengths of the various gauge fields belonging to the vector or to the Weyl supermultiplet comprise a chiral multiplet. Such multiplets are described in superspace by chiral superfields. At this point it is convenient to systematize our discussion by using superspace notions, although we do not intend to make an essential use of superfields. Scalar chiral fields in $N = 2$ superspace have $16 + 16$ bosonic + fermionic field components, but there exists a

constraint which reduces the superfield to only $8 + 8$ field components. This constraint expresses higher- θ components of the superfield in terms of lower- θ components or space-time derivatives thereof. The vector supermultiplet and the Weyl supermultiplet are both related to *reduced* chiral multiplets. Note, however, that products of reduced chiral superfields constitute general chiral fields.

The vector supermultiplet is described by a scalar reduced chiral superfields, whose lowest- θ component is a complex field which we denote by X . Then there is a doublet of chiral fermions Ω_i , where i is an $SU(2)$ R-symmetry doublet index. The position of the index i indicates the chirality of the spinor field: Ω_i carries positive, and Ω^i negative chirality. The fields \bar{X} and Ω^i appear as lowest- θ components in the anti-chiral superfield that one obtains by complex conjugation of the chiral superfield. We recall that the so-called R-symmetry group is defined as the maximal group that rotates the supercharges in a way that commutes with Lorentz symmetry and is compatible with the supersymmetry algebra. For $N = 2$ supersymmetry this group equals $SU(2) \times U(1)$, which acts chirally on the spinors. In spite of its name, R-symmetry does not necessarily constitute an invariance of supersymmetric Lagrangians. Finally, at the θ^2 -level, we encounter the field strength $F_{\mu\nu}$ of the gauge field and an auxiliary field which we write as a symmetric real tensor, $Y_{ij} = Y_{ji} = \varepsilon_{ik}\varepsilon_{jl}Y^{kl}$. Here we note that complex conjugation will often be indicated by raising and/or lowering of $SU(2)$ indices. One can easily verify that in this way we have precisely $8 + 8$ independent field components (which we will refer to as off-shell degrees of freedom, as we did not impose any field equations). Note the difference with on-shell degrees of freedom. The conventional Lagrangian for the vector supermultiplet describes $4 + 4$ physical massless bosonic and fermionic states: 2 scalars associated with X and \bar{X} , the 2 helicities associated with the vector gauge field, and 2 Majorana fermions, each carrying 2 helicities.

The Weyl supermultiplet has a rather similar decomposition, but in this case the reduced chiral superfield is not a scalar but an anti-selfdual

Lorentz tensor. For extended supersymmetry the Weyl superfield is also assigned to the antisymmetric representation of the R-symmetry group, so that its lowest-order θ -component is denoted by $T_{ab}{}^{ij}$. Its complex conjugate belongs to the corresponding anti-chiral superfield and its corresponding tensor is selfdual and denoted by T_{abij} . Here the indices a, b, \dots denote the components of space-time tangent space tensors. In view of its tensorial character the Weyl supermultiplet comprises $24 + 24$ off-shell degrees of freedom. The covariant components of the Weyl multiplet are as follows. Linear in θ one has the fermions decomposing into the field strength of the gravitini and a doublet spinor χ^i . The gravitino field strengths comprise 16 degrees of freedom so that together with the spinors χ^i we count 24 off-shell degrees of freedom. At the θ^2 -level we have the Weyl tensor, the field strengths belonging to the gauge fields associated with R-symmetry, and a real scalar field denoted by D , comprising $5, 4 \times 3$ and 1 off-shell degrees of freedom, respectively. Together with the 6 degrees of freedom belonging to $T_{ab}{}^{ij}$, we count 24 bosonic degrees of freedom.

The Weyl multiplet contains the fields of $N = 2$ conformal supergravity and an invariant action can be written down that is quadratic in its components. Although $T_{ab}{}^{ij}$ is not subject to a Bianchi identity, it is often called the “graviphoton field strength”. The reason for this misnomer is that the gravitini transform into $T_{ab}{}^{ij}$ and in locally supersymmetric Lagrangians of vector multiplets that are at most quadratic in space-time derivatives, $T_{ab}{}^{ij}$ acts as an auxiliary field and couples to a field-dependent linear combination of the vector multiplet field strengths. For this class of Lagrangians, all the fields of the Weyl multiplet with the exception of the graviton and the gravitini fields, act as auxiliary fields.

In this subsection we mainly describe linearized results. Ultimately we are interested in constructing a theory of local supersymmetry. This means that the vector multiplets must first be formulated in a supergravity background. This leads to additional terms in the supersymmetry transformation rules and in the superfield components which depend on the supergravity background. Some of these terms correspond to re-

placing ordinary space-time derivatives by covariant ones. However, we consider only vector multiplets and the Weyl multiplet here and the latter describes the supermultiplet of *conformal* supergravity. Consistency therefore requires that we formulate the vector supermultiplet in a superconformal background, and, indeed, the vector supermultiplet is a representation of the full superconformal algebra. Therefore all the superconformal symmetries can be realized as local symmetries. Naturally the Weyl multiplet itself will also acquire non-linear corrections but those do not involve the vector multiplet fields. The reader may wonder why we are interested in conformal supergravity here, but this will be clarified shortly. Other than that, the situation is conceptually the same as when considering multiplets of matter fields coupled to a nonabelian gauge theory, or to gravity.

5.2. Supersymmetric actions

In view of the fact that both the vector and Weyl supermultiplets are described by chiral superfields, even beyond their linearized form, it is clear how to construct supersymmetric actions. Namely one takes some function of the vector superfield (actually we will need several of these superfields, which we label by indices $I = 0, 1, 2, \dots, n$) and the square of the Weyl superfield. Taking the square implies no loss of generality because we are interested in Lorentz invariant couplings. When expanding the superfields in θ components, one generates multiple derivatives of this function which depend on the lowest- θ components, X and

$$A = (T_{ab}{}^{ij} \varepsilon_{ij})^2. \quad (27)$$

Because the function is holomorphic, i.e., it depends on X and A , but not on their complex conjugates, we take the imaginary part of the resulting expression. However, in order that the action is superconformally invariant, the function $F(X, A)$ must be holomorphic and homogeneous,

$$F(\lambda X, \lambda^2 A) = \lambda^2 F(X, A). \quad (28)$$

We refrain from giving full results. In principle they are derived straightforwardly, but the for-

mulae are often lengthy and require extra definitions. Therefore we discuss only a few characteristic terms.

First of all, let us consider the scalar kinetic terms. They are accompanied by a coupling to the Ricci scalar and the scalar field D of the Weyl multiplet in the following way,

$$\begin{aligned} \mathcal{L} \propto & i(\partial_\mu F_I \partial^\mu \bar{X}^I - \partial_\mu \bar{F}_I \partial^\mu X^I) \\ & - i(\tfrac{1}{6}R - D)(F_I \bar{X}^I - \bar{F}_I X^I), \end{aligned} \quad (29)$$

where F_I denotes the derivative of F with respect to X^I . Observe that when the function F depends on $T_{ab}{}^{ij}$ this will generate interactions between the kinetic term for the vector multiplet scalars and the tensor field of the Weyl multiplet. Of course, this pattern continues for other terms.

A second term concerns the kinetic term of the vector fields, which is proportional to the second derivative of the function F ,

$$\begin{aligned} \mathcal{L} \propto & \tfrac{1}{4} i F_{IJ} (F_{ab}^{-I} - \tfrac{1}{4} \bar{X}^I T_{ab}{}^{ij} \varepsilon_{ij}) \\ & \times (F^{-abJ} - \tfrac{1}{4} \bar{X}^J T^{abkl} \varepsilon_{kl}) + \text{h.c.} \end{aligned} \quad (30)$$

A third term involves the square of the Weyl tensor, contained in the tensor $\mathcal{R}(M)$,

$$\begin{aligned} \mathcal{L} \propto & 16 i F_A [2 \mathcal{R}(M)^{cd}{}_{ab} \mathcal{R}(M)^{-ab}{}_{cd} \\ & - 16 T^{abij} D_a D^c T_{cbij}] + \text{h.c.} \end{aligned} \quad (31)$$

Here D_a denotes a superconformally covariant derivative (which also contains terms proportional to the Ricci tensor). We refrain from giving further details at this point and refer to the literature.

5.3. Compensator multiplets

The theories discussed so far are invariant under the local symmetries of the superconformal algebra. This high degree of symmetry seems unnecessary for, or even an obstacle to, practical applications. The purpose of this section is to explain that this is not the case.

Let us start with a simple example, namely massive $SU(N)$ Yang-Mills theory, with Lagrangian,

$$\begin{aligned} \mathcal{L} = & \tfrac{1}{4} \text{Tr}[F_{\mu\nu}(W) F^{\mu\nu}(W)] \\ & + \tfrac{1}{2} M^2 \text{Tr}[W_\mu W^\mu], \end{aligned} \quad (32)$$

where we use a Lie-algebra valued notation and the definition $F_{\mu\nu} = 2\partial_{[\mu}W_{\nu]} - [W_\mu, W_\nu]$. Introduce now a matrix-valued field $\Phi \in \text{SU}(N)$ transforming under $\text{SU}(N)$ gauge transformations from the left and substitute $W_\mu = \Phi^{-1}D_\mu\Phi$ into the Lagrangian, where the covariant derivative reads $D_\mu\Phi = (\partial_\mu - A_\mu)\Phi$. The first term is not affected by this transformation which takes the form of a field-dependent gauge transformation. But the mass term changes, and we find the following Lagrangian,

$$\mathcal{L} = \frac{1}{4}\text{Tr}[F_{\mu\nu}(A)F^{\mu\nu}(A)] - \frac{1}{2}M^2\text{Tr}[D_\mu\Phi D^\mu\Phi^{-1}], \quad (33)$$

which is manifestly gauge invariant. Clearly, this is a massless gauge theory coupled to $N^2 - 1$ scalars. However, this formulation is gauge equivalent to (32), as one readily verifies by imposing the gauge condition $\Phi = \mathbf{1}$.

What do we achieve by rewriting (32) into the form (33)? Both Lagrangians describe the same number of physical states and are based on the same number of off-shell degrees of freedom. In (32) the degrees of freedom are contained in a single field, W_μ , carrying 4 components per generator. In (33), however, the degrees of freedom are distributed over two fields in a local and Lorentz invariant way, namely 3 components per generator in A_μ (we subtracted the gauge degrees of freedom) and 1 component per generator in Φ . Hence the second formulation can be regarded as reducible, and this reducibility has been achieved by introducing extra gauge invariance.

A similar situation exists for gravity, as is shown by the Lagrangian,

$$\sqrt{|g|}\mathcal{L} \propto \sqrt{|g|} [g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{6}R\phi^2], \quad (34)$$

which is invariant under local scale transformation with parameter $\Lambda(x)$: $\delta\phi = \Lambda\phi$, $\delta g_{\mu\nu} = -2\Lambda g_{\mu\nu}$. This Lagrangian is gauge equivalent to the Einstein-Hilbert Lagrangian. To see this one either rewrites it in terms of a scale invariant metric $\phi^2 g_{\mu\nu}$, or one simply imposes the gauge condition and sets ϕ equal to a constant (which will then be related to Newton's constant). Again the Einstein-Hilbert Lagrangian and (34) contain the same number of off-shell degrees of freedom,

but the latter field configuration is reducible: $g_{\mu\nu}$ describes only 5 degrees of freedom (in view of the local scale invariance) and the sixth one can be assigned to the scalar field ϕ .

Fields such as Φ and ϕ are called compensating fields, because they can be used to convert any quantity that transforms under the gauge symmetry into a gauge invariant one. Often the gauge equivalent formulation, based on the introduction of compensator fields and gauge symmetries at the same time, is exploited for reasons of renormalizability as one can use the gauge freedom to choose a different gauge that leads to better short-distance behaviour. This is not the issue here but the crucial point is that the compensating degrees of freedom must be contained in full supermultiplets. By keeping the gauge invariance manifest one realizes a higher degree of symmetry which facilitates the construction of Lagrangians and clarifies the geometrical features of the resulting supergravity theories. In this way, pure $N = 2$ supergravity is, for instance, constructed from two compensating supermultiplets, one of which is a vector multiplet and the other one can be a tensor multiplet or a hypermultiplet. Of these two supermultiplet only the vector field will describe physical degrees of freedom (namely, those corresponding to the graviphoton). The other components play the role of either compensating fields (associated with local scale, R-symmetry and special conformal supersymmetry transformations), or are constrained by the field equations, either by Lagrange multipliers, or because they are auxiliary.

6. BPS ATTRACTORS

As we already discussed in section 4 the BPS attractor equations follow from the requirement of full $N = 2$ supersymmetry at the horizon. In the context of an off-shell representation of supersymmetry, the corresponding equations can be derived in a way that is rather independent of the action. As explained in the previous section, $N = 2$ vector multiplets contain complex physical scalar fields which we denoted by X^I . In the context of the superconformal framework these fields are defined projectively, in view of the invariance

under local scale and $U(1)$ transformations. The action for vector multiplets with additional interactions involving the square of the Weyl tensor is encoded in a holomorphic function $F(X, A)$, which is homogenous of second degree (c.f. (28)). Here A is quadratic in the anti-selfdual field $T_{ab}{}^{ij}$, as shown in (27). The normalizations of the Lagrangian and of the charges adopted below differ from the normalizations used in section 4.

Another issue that we should explain concerns electric/magnetic duality, which in principle pertains to the gauge fields. Straightforward application of this duality to an $N = 2$ supersymmetric Lagrangian with vector multiplets, leads to a new Lagrangian that no longer takes the canonical form in terms of a function $F(X, A)$. In order to bring it into that form one must simultaneously apply a field redefinition to the scalar and spinor fields. On the scalar fields, this redefinition follows from the observation that $(X^I, F_I(X, A))$ transforms as a symplectic vector analogous to the tensors $(F_{\mu\nu}{}^I, G_{\mu\nu I})$ discussed previously. Hence the $(2n + 2)$ -dimensional vector $(X^I, F_I(X, A))$ is rotated into a new vector $(\tilde{X}^I, \tilde{F}_I)$. When this rotation belongs to $Sp(2n + 2; \mathbb{R})$ then \tilde{F}_I can be written as the derivative of a new function $\tilde{F}(\tilde{X}, A)$ with respect to \tilde{X}^I . The new function then encodes the dual action. The need for this additional field redefinition follows from the fact that the gauge fields, and thus their field strengths $F_{\mu\nu}{}^I$, and the fields X^I have a well-defined relation encoded in the supersymmetry transformation rules. Therefore, the transformations of $(X^I, F_I(X, A))$ must be taken into account when considering electric/magnetic duality. We refer to [24, 28] for further details and a convenient list of formulae.

To determine the BPS attractor equations one classifies all possible $N = 2$ supersymmetric solutions. This is done by studying the supersymmetry variations of the fermions in an arbitrary bosonic background. Requiring that these variations vanish then imposes strong restrictions on this background. The analysis was performed in [15] for $N = 2$ supergravity with an arbitrary number of vector multiplets and hypermultiplets, including higher-order derivative couplings proportional to the square of the Weyl tensor. Also

the interpolating BPS solutions were studied in considerable detail. It was found that $N = 2$ supersymmetric solutions are unique and depend on a harmonic function with a single center. Hence the horizon geometry and the values of the relevant fields are fully determined in terms of the charges. The hypermultiplet scalar fields are covariantly constant but otherwise arbitrary. However, the horizon and the entropy do not depend on these fields, so that they can be ignored. In the absence of charges one is left with flat Minkowski space-time with arbitrary constant moduli and $T_{ab}{}^{ij} = 0$.

As it turns out the attractor equations have a universal form. Before commenting further on their derivation, let us present the equations, which are manifestly covariant under electric/magnetic duality,

$$\mathcal{P}^I = 0, \quad \mathcal{Q}_I = 0, \quad \Upsilon = -64, \quad (35)$$

where

$$\begin{aligned} \mathcal{P}^I &\equiv p^I + i(Y^I - \bar{Y}^I), \\ \mathcal{Q}_I &\equiv q_I + i(F_I - \bar{F}_I). \end{aligned} \quad (36)$$

Here the Y^I and Υ are related to the X^I and A , respectively, by a uniform rescaling and F_I and F_{Υ} will denote the derivatives of $F(Y, \Upsilon)$ with respect to Y^I and Υ . To explain the details of the rescaling, we introduce the complex quantity Z , sometimes referred to as the 'holomorphic BPS mass', which equals the central charge associated with the vector supermultiplets. In terms of the original variables X^I it is defined as

$$Z = \exp[\mathcal{K}/2] (p^I F_I(X, A) - q_I X^I), \quad (37)$$

where

$$e^{-\mathcal{K}} = i(\bar{X}^I F_I(X, A) - \bar{F}_I(\bar{X}, \bar{A}) X^I). \quad (38)$$

At the horizon the variables Y^I and Υ are defined by

$$\begin{aligned} Y^I &= \exp[\mathcal{K}/2] \bar{Z} X^I, \\ \Upsilon &= \exp[\mathcal{K}] \bar{Z}^2 A. \end{aligned} \quad (39)$$

Note that Y^I and Υ are invariant under arbitrary complex rescalings of the underlying variables X^I

and A . The reader may easily verify that for fields satisfying the attractor equations (35), one establishes that

$$|Z|^2 \equiv p^I F_I - q_I Y^I, \quad (40)$$

which is obviously real and positive, is equal to $i(\bar{Y}^I F_I - Y^I \bar{F}_I)$.

Finally we wish to draw attention to just one aspect of the derivation of the attractor equations (35). Consider the spinor fields belonging to the vector supermultiplets, and concentrate on their supersymmetry variation in terms of a two-rank tensor,

$$\delta\Omega_i{}^I = \frac{1}{2}\varepsilon_{ij}\gamma^{ab}\epsilon^j\left(F_{ab}^{-I} - \frac{1}{4}\varepsilon_{kl}T_{ab}{}^{kl}\bar{X}^I\right). \quad (41)$$

This particular linear combination of the field strength F_{ab}^{-I} and the field \bar{X}^I arises because the symmetry transformations are evaluated in a superconformal background. Full supersymmetry therefore implies that the right-hand side of (41) vanishes, so that

$$F_{ab}^{-I} = \frac{1}{4}\varepsilon_{kl}T_{ab}{}^{kl}\bar{X}^I. \quad (42)$$

An extension of this argument gives a similar result for the conjugate field strengths,

$$G_{abI}^- = \frac{1}{4}\varepsilon_{kl}T_{ab}{}^{kl}\bar{F}_I. \quad (43)$$

Note that these two equations are consistent with respect to electric/magnetic duality. Given the fact that the field strengths F_{ab}^I and G_{abI} satisfy the Maxwell equations and therefore become proportional to the magnetic and electric charges, p^I and q_I , it is not surprising that one finds the attractor equations (35). For further details of the analysis we refer to [15].

6.1. The BPS entropy function

The BPS attractor equations follow also from a variational principle based on the entropy function [17,18],

$$\Sigma(Y, \bar{Y}, p, q) = \mathcal{F}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) - q_I(Y^I + \bar{Y}^I) + p^I(F_I + \bar{F}_I), \quad (44)$$

where p^I and q_I couple to the corresponding magneto- and electrostatic potentials at the

horizon (c.f. [15]) in a way that is consistent with electric/magnetic duality. The quantity $\mathcal{F}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$, which will be regarded as a ‘free energy’ in what follows, is defined by

$$\mathcal{F}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) = -i(\bar{Y}^I F_I - Y^I \bar{F}_I) - 2i(\Upsilon F_\Upsilon - \bar{\Upsilon} \bar{F}_\Upsilon), \quad (45)$$

where $F_\Upsilon = \partial F / \partial \Upsilon$. Just as the entropy function discussed in section 4, the entropy function (44) transforms as a function under electric/magnetic duality [28]. Varying this entropy function with respect to the Y^I , while keeping the charges and Υ fixed, yields the result,

$$\delta\Sigma = \mathcal{P}^I \delta(F_I + \bar{F}_I) - \mathcal{Q}_I \delta(Y^I + \bar{Y}^I). \quad (46)$$

Here we made use of the homogeneity of the function $F(Y, \Upsilon)$. Under the mild assumption that the matrix

$$N_{IJ} = i(\bar{F}_{IJ} - F_{IJ}), \quad (47)$$

is non-degenerate, it thus follows that stationary points of Σ satisfy the attractor equations. The macroscopic entropy is equal to the entropy function taken at the attractor point. This implies that the macroscopic entropy is the Legendre transform of the free energy (45). An explicit calculation yields the entropy formula [29,30,31],

$$\begin{aligned} \mathcal{S}_{\text{macro}}(p, q) &= \pi \Sigma \Big|_{\text{attractor}} \\ &= \pi \left[|Z|^2 - 256 \text{Im} F_\Upsilon \right]_{\Upsilon=-64}. \end{aligned} \quad (48)$$

Here the first term represents a quarter of the horizon area (in Planck units) so that the second term defines the deviation from the Bekenstein-Hawking area law. The entropy coincides precisely with the Wald entropy [16] as given by the right-hand side of (26). In fact, the original derivation of (48) was not based on an entropy function and made direct use of the expression (26).

In the absence of Υ -dependent terms, the homogeneity of the function $F(Y)$ implies that the area scales quadratically with the charges, as was discussed already at the beginning of section 2. However, in view of the fact that Υ takes a fixed

value, the second term will be subleading in the limit of large charges.² Note, however, that also the area will contain subleading terms, as it depends on Υ .

The entropy equation (48) has been confronted with the result of microstate counting, for instance, in the situation described in section 3. In that case the effective supergravity action is known and based on the function

$$F(Y, \Upsilon) = -\frac{1}{6} \frac{C_{ABC} Y^A Y^B Y^C}{Y^0} - \frac{c_{2A}}{24 \cdot 64} \frac{Y^A}{Y^0} \Upsilon. \quad (49)$$

Substituting this result into (48) and imposing the attractor equations (35) with $p^0 = 0$, one indeed derives the result (13) for the macroscopic entropy [29]. The entropy formula (48) has also been put to a test in other cases. Some of them will be discussed in due course.

The relation between (44) and the entropy function introduced in section 4 was discussed in [25], where it was established that both entropy functions lead to identical results for BPS black holes. When the black holes are not BPS (i.e. have no residual supersymmetry), then the entropy function (44) is simply not applicable. In this connection the question arises whether other, independent, higher-derivative interactions associated with matter multiplets will not contribute to the entropy either. For instance, Lagrangians for tensor supermultiplets that contain interactions of fourth order in space-time derivatives, lead to terms quadratic in the Ricci scalar that will in principle contribute to the Wald entropy [32]. Indeed, for non-BPS black holes these terms yield finite contributions to the entropy, but for BPS black holes these corrections vanish. A comprehensive treatment of higher-derivative interactions is yet to be given for $N = 2$ supergravity, but it seems that this result is generic. At any rate, this observation seems in line with more recent findings [33,34] based on heterotic string α' -corrections encoded in a higher-derivative effective action in higher

dimensions. In four dimensions this action leads to additional matter-coupled higher-derivative interactions. When these are taken into account, the matching of the macroscopic entropy with the microscopic result is established [34].

Another modification concerns possible non-holomorphic corrections to the function $F(X, A)$. This holomorphic function leads to a supersymmetric action that corresponds to the so-called effective Wilsonian action, based on integrating out the massive degrees of freedom. The Wilsonian action describes the correct physics for energies between appropriately chosen infrared and ultraviolet cutoffs. However, this action does not reflect all the physical symmetries. To preserve those symmetries non-holomorphic contributions should be included associated with integrating out massless degrees of freedom. In the special case of heterotic black holes in $N = 4$ supersymmetric compactifications, the requirement of explicit S-duality invariance of the entropy and the attractor equations allows one to determine the contribution from these non-holomorphic corrections, as was first demonstrated in [31] for BPS black holes. In [35,18] it was established that non-holomorphic corrections to the BPS entropy function (44) can be encoded into a real function $\Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$ which is homogeneous of second degree. The modifications to the entropy function are then effected by substituting $F(Y, \Upsilon) \rightarrow F(Y, \Upsilon) + 2i\Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$. There are good reasons to expect that this same substitution should be applied to the more general entropy function based on (23) [25]. Note that when Ω is harmonic, i.e., when it satisfies $\partial^2 \Omega / \partial Y^I \partial \bar{Y}^J = 0$, it can simply be absorbed into the original holomorphic function $F(Y, \Upsilon)$.

Finally we point out the existence of a formulation in terms of real, rather than the complex fields (Y^I, Υ) , that we used before. This formulation is manifestly covariant with respect to electric/magnetic duality. We first decompose Y^I and F_I into their real and imaginary parts,

$$Y^I = x^I + iu^I, \quad F_I = y_I + iv_I, \quad (50)$$

where $F_I = F_I(Y, \Upsilon)$. The real parametrization is obtained by taking $(x^I, y_I, \Upsilon, \bar{\Upsilon})$ instead of $(Y^I, \bar{Y}^I, \Upsilon, \bar{\Upsilon})$ as the independent variables.

² Here one usually assumes that $F(Y, \Upsilon)$ can be expanded in positive powers of Υ .

This reparametrization is only well defined provided $\det(N_{IJ}) \neq 0$. Subsequently one defines the Hesse potential, the real analogue of the Kähler potential, which equals twice the Legendre transform of the imaginary part of the prepotential with respect to $u^I = \text{Im } Y^I$,

$$\mathcal{H}(x, y, \Upsilon, \bar{\Upsilon}) = 2 \text{Im } F(x + iu, \Upsilon, \bar{\Upsilon}) - 2 y_I u^I, \quad (51)$$

Owing to the homogeneity of the function $F(Y, \Upsilon)$ one can show that the free energy (45) equals twice the Hesse potential. The entropy function (44) is now replaced by

$$\Sigma(x, y, p, q) = 2 \mathcal{H}(x, y, \Upsilon, \bar{\Upsilon}) - 2 q_I x^I + 2 p^I y_I, \quad (52)$$

and it is straightforward to show that the extremization equations are just the attractor equations (35), expressed in terms of the new variables (x^I, y_I) . The value of $\Sigma(x, y, p, q)$ at the attractor point coincides again with the macroscopic entropy.

6.2. Partial Legendre transforms

It is, of course, possible to define the macroscopic entropy as a Legendre transform with respect to only a subset of the fields, by substituting part of the attractor equations such that the variational principle remains valid. These partial Legendre transforms constitute a hierarchy of Legendre transforms for the black hole entropy. Here we discuss two relevant examples, namely, the one proposed in [36] where all the magnetic attractor equations are imposed, and the dilatonic one for heterotic black holes, where only two real potentials are left which together constitute the complex dilaton field [31]. A possible disadvantage of considering partial Legendre transforms is that certain invariances may no longer be manifest. As it turns out, the dilatonic formulation does not suffer from this shortcoming. Apart from that, there is no reason to prefer one version over the other. This will change in section 7 when we discuss corresponding partition functions and inverse Laplace transforms for the microscopic degeneracies in semiclassical approximation.

Let us first impose the magnetic attractor equations so that only the real parts of the Y^I will remain as independent variables. Hence one makes the substitution,

$$Y^I = \frac{1}{2}(\phi^I + i p^I). \quad (53)$$

The entropy function (44) then takes the form,

$$\Sigma(\phi, p, q) = \mathcal{F}_E(p, \phi, \Upsilon, \bar{\Upsilon}) - q_I \phi^I, \quad (54)$$

where the corresponding free energy $\mathcal{F}_E(p, \phi)$ equals

$$\mathcal{F}_E(p, \phi) = 4 \left[\text{Im } F(Y, \Upsilon) + \Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) \right]_{Y^I = (\phi^I + i p^I)/2}. \quad (55)$$

To derive this result one makes use of the homogeneity of the functions $F(Y, \Upsilon)$ and $\Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$. The latter function may contain possible non-holomorphic terms. When extremizing (55) with respect to ϕ^I we obtain the attractor equations $q_I = \partial \mathcal{F}_E / \partial \phi^I$. This shows that the macroscopic entropy is a Legendre transform of $\mathcal{F}_E(p, \phi)$ subject to $\Upsilon = -64$, as was first noted in [36] in the absence of Ω . In the latter case this Legendre transform led to the conjecture that there is a relation with topological strings, in view of the fact that $\exp[\mathcal{F}_E]$ equals the modulus square of the topological string partition function [37]. We return to this in subsection 7.1.

Along the same line one can now proceed and eliminate some of the ϕ^I as well. A specific example of this is the dilatonic formulation heterotic black holes, where we eliminate all the ϕ^I with the exception of two of them which parametrize the complex dilaton field. This leads to an entropy function that depends only on the charges and on the dilaton field [31, 35]. Here it is convenient to include all the Υ -dependent terms into Ω , which also contains the non-holomorphic corrections. The heterotic classical function $F(Y)$ is given by

$$F(Y) = - \frac{Y^1 Y^a \eta_{ab} Y^b}{Y^0}, \quad a = 2, \dots, n, \quad (56)$$

with real constants η_{ab} . In the application that we will be considering the function Ω depends only

linearly on Υ and $\bar{\Upsilon}$, as well as on the dilaton field $S = -iY^1/Y^0$ and its complex conjugate \bar{S} . The result for the BPS entropy function then takes the form,

$$\Sigma(S, \bar{S}, p, q) = -\frac{q^2 - ip \cdot q (S - \bar{S}) + p^2 |S|^2}{S + \bar{S}} + 4\Omega(S, \bar{S}, \Upsilon, \bar{\Upsilon}), \quad (57)$$

where q^2 , p^2 and $p \cdot q$ are T-duality invariant bilinears of the various charges, defined by

$$\begin{aligned} q^2 &= 2q_0 p^1 - \frac{1}{2} q_a \eta^{ab} q_b, \\ p^2 &= -2p^0 q_1 - 2p^a \eta_{ab} p^b, \\ q \cdot p &= q_0 p^0 - q_1 p^1 + q_a p^a. \end{aligned} \quad (58)$$

Note that these bilinears are not positive definite. Furthermore, Ω captures the Υ -dependent corrections to the classical result (56). Its form was derived for $N = 4$ heterotic string compactifications by requiring S-duality of the attractor equations and of the entropy [31,35],

$$\begin{aligned} \Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) &= \\ \frac{1}{256\pi} &[\Upsilon \log \eta^{24}(S) + \bar{\Upsilon} \log \eta^{24}(\bar{S}) \\ &+ \frac{1}{2}(\Upsilon + \bar{\Upsilon}) \log(S + \bar{S})^{12}], \end{aligned} \quad (59)$$

where $\eta(S)$ denotes the Dedekind function. Note the presence of the last term which is non-holomorphic. This term is in accord with the result for the effective action obtained from five-brane instantons [38]. The attractor equations associated with the dilaton take the form $\partial_S \Sigma(S, \bar{S}, p, q) = 0$.

It is interesting to consider the consequences of (57) in the classical case ($\Omega = 0$). Then the attractor equations yield the following values for the real part of the dilaton field and the macroscopic entropy,

$$\begin{aligned} S + \bar{S} &= -\frac{2}{p^2} \sqrt{p^2 q^2 - (p \cdot q)^2}, \\ \mathcal{S}_{\text{macro}} &= \pi \sqrt{p^2 q^2 - (p \cdot q)^2}. \end{aligned} \quad (60)$$

Obviously there are two types of black holes, depending on whether $p^2 q^2 - (p \cdot q)^2$ is positive or zero. In the context of $N = 4$ heterotic string compactifications these correspond to 1/4- and

1/2-BPS black holes, respectively. The 1/4-BPS states are dyonic, so that they necessarily carry both electric and magnetic charges. The 1/2-BPS states can be purely electric. We derived these results from the BPS entropy function, but they have also been obtained directly from the full supergravity solutions [39,40,41].

Clearly the 1/4-BPS black holes are *large* black holes as their area (entropy) is nonzero and scales quadratically with the charges. Note that, depending on the choice of the charges, the complex dilaton field can remain *finite* in the limit of large charges. This is relevant when studying the asymptotic growth of the dyonic degeneracy of 1/4-BPS dyons in heterotic string theory compactified on a six-torus and in the related class of heterotic CHL models [42]. These degeneracies are encoded in automorphic forms $\Phi_k(\rho, \sigma, v)$ of weight k under $\text{Sp}(2, \mathbb{Z})$, or an appropriate subgroup thereof [43,44]. The torus compactification corresponds to $k = 10$ and the CHL models to $k = 1, 2, 4, 6$. The three modular parameters, ρ, σ, v , parametrize the period matrix of an auxiliary genus-two Riemann surface, which takes the form of a complex, symmetric, two-by-two matrix. The microscopic degeneracy of 1/4-BPS dyons is expressed as an integral over an appropriate 3-cycle,

$$d_k(p, q) = \oint d\rho d\sigma dv \frac{e^{i\pi[\rho p^2 + \sigma q^2 + (2v-1)p \cdot q]}}{\Phi_k(\rho, \sigma, v)}. \quad (61)$$

Since Φ_k has zeros in the interior of the Siegel half-space in addition to the zeros at the cusps, the value of the integral (61) depends sensitively on the choice of the integration 3-cycles. The charges are in general integer, with the exception of q_1 which equals a multiple of N , and p^1 which is fractional and quantized in units of $1/N$. Here N and k are related by $(k+2)(N+1) = 24$. The inverse of the modular form Φ_k takes the form of a Fourier sum with integer powers of $\exp[2\pi i \rho]$ and $\exp[2\pi i v]$ and fractional powers of $\exp[2\pi i \sigma]$ which are multiples of $1/N$. The 3-cycle is then defined by choosing integration contours where the real parts of ρ and v take values in the interval $(0, 1)$ and the real part of σ takes values in the

interval $(0, N)$. The formula (61) is manifestly invariant under T-duality (the integrand depends on the three T-duality invariant bilinears (58)), as well as under S-duality, which is a subgroup of the full modular group.

The integral (61) can be evaluated in saddle-point approximation which yields the leading and subleading contributions to $d_k(p, q)$ [35, 44]. As it turns out these contributions are precisely encoded in (57) and (59), including the Dedekind eta-functions and the non-holomorphic terms. The presence of the non-holomorphic terms is not surprising in view of the S-duality invariance. Note that the expression (59) refers to $k = 10$ and that there exist similar formulae for $k = 1, 2, 4, 6$.

The 1/2-BPS black holes are *small* black holes as their area scales linearly with the charges. According to (60) their entropy (and horizon area) vanishes while the dilaton field diverges at the horizon, because we have $p^2 = p \cdot q = 0$. To describe the situation more accurately one retains the leading term of (59). In that case one obtains the following result (we restrict ourselves to $k = 10$) [31],

$$\begin{aligned} S + \bar{S} &\approx \sqrt{|q^2|/2}, \\ \mathcal{S}_{\text{macro}} &\approx 4\pi \sqrt{|q^2|/2} - 6 \log |q^2|, \end{aligned} \quad (62)$$

where the logarithmic term is due to the non-holomorphic contribution. Because the dilaton is large in this case, all the exponentials in the Dedekind eta-function are suppressed and we are at weak string coupling $g_s \propto (S + \bar{S})^{-1/2}$.

We already stressed that small black holes have a size of the order of the string scale and, indeed, these states are precisely generated by perturbative heterotic string states arising in $N = 4$ supersymmetric compactifications to four space-time dimensions. In the supersymmetric right-moving sector these states carry only momentum and winding and contain no oscillations, whereas in the left-moving sector oscillations are allowed that satisfy the string matching condition. The oscillator number is then linearly related to q^2 . These perturbative states received quite some attention in the past [45]. Because the higher-mass string states are expected to be within their Schwarzschild radius, it was con-

jectured that they should have an interpretation as black holes.³ Their calculable level density, proportional to the exponent of $4\pi\sqrt{|q^2|/2}$, implies a nonzero microscopic entropy for these black holes [46].

This result was confronted with explicit black hole solutions [47, 48, 49] based on standard supergravity Lagrangians that are at most quadratic in derivatives, which have a vanishing horizon area. Based on the area law one thus obtains a vanishing macroscopic entropy. The fact that (48), which takes into account higher-derivative interactions, can nicely account for the discrepancies encountered in the classical description of the 1/2-BPS black holes, was first emphasized in [50, 51]. Note also that, since the electric states correspond to perturbative heterotic string states, their degeneracy is known from string theory and given by

$$\begin{aligned} d(q) &= \oint d\sigma \frac{e^{i\pi\sigma q^2}}{\eta^{24}(\sigma)} \\ &\approx \exp\left(4\pi\sqrt{|q^2|/2} - \frac{27}{4}\log|q^2|\right), \end{aligned} \quad (63)$$

where the integration contour encircles the point $\exp(2\pi i\sigma) = 0$. This large- $|q^2|$ approximation is based on a standard saddle-point approximation. Obviously the leading term of (63) is in agreement with (62, but beyond that there is a disagreement as the logarithmic corrections carry different coefficients. This discrepancy may be regarded as a first indication that small black holes are not well understood (for a discussion, see, for instance, [18]). Therefore we will mainly concentrate on large black holes in the next chapter.

7. PARTITION FUNCTIONS AND INVERSE LAPLACE TRANSFORMS

To again make the connection with microstate degeneracies, we conjecture, in the spirit of [36], that the Legendre transforms of the entropy are indicative of a thermodynamic origin of the various entropy functions. It is then natural to assume that the corresponding free energies are

³ The idea that elementary particles, or string states, are behaving like black holes, has been around for quite some time.

related to black hole partition functions corresponding to suitable ensembles of black hole microstates. Following [18], we define

$$Z(\phi, \chi) = \sum_{\{p, q\}} d(p, q) e^{\pi[q_I \phi^I - p^I \chi_I]}, \quad (64)$$

where $d(p, q)$ denotes the microscopic degeneracies of the black hole microstates with black hole charges p^I and q_I . This is the partition sum over a canonical ensemble, which is invariant under the various duality symmetries, provided that the electro- and magnetostatic potentials (ϕ^I, χ_I) transform as a symplectic vector. Identifying a free energy with the logarithm of $Z(\phi, \chi)$, it is clear that it should, perhaps in an appropriate limit, be related to the macroscopic free energy introduced earlier. On the other hand, viewing $Z(\phi, \chi)$ as an analytic function in ϕ^I and χ_I , the degeneracies $d(p, q)$ can be retrieved by an inverse Laplace transform,

$$d(p, q) \propto \int d\chi_I d\phi^I Z(\phi, \chi) e^{\pi[-q_I \phi^I + p^I \chi_I]}, \quad (65)$$

where the integration contours run, for instance, over the intervals $(\phi - i, \phi + i)$ and $(\chi - i, \chi + i)$ (we are assuming an integer-valued charge lattice). Obviously, this makes sense as $Z(\phi, \chi)$ is formally periodic under shifts of ϕ and χ by multiples of $2i$.

These arguments suggest to identify $Z(\phi, \chi)$ with the Hesse potential (51),

$$\sum_{\{p, q\}} d(p, q) e^{\pi[q_I \phi^I - p^I \chi_I]} \sim \sum_{\text{shifts}} e^{2\pi \mathcal{H}(\phi/2, \chi/2, \Upsilon, \bar{\Upsilon})}, \quad (66)$$

where Υ is equal to its attractor value and where we suppressed possible non-holomorphic contributions for simplicity. However, the Hesse potential is a macroscopic quantity which does not in general exhibit the periodicity that is characteristic for the partition function. Therefore, the right-hand side of (66) requires an explicit periodicity sum over discrete imaginary shifts of the ϕ

and χ .⁴ When substituting $2\pi\mathcal{H}$ into the inverse Laplace transform, we expect that the periodicity sum can be incorporated into the integration contour.

It is in general difficult to find an explicit representation for the Hesse potential, as the relation (50) between the complex variables Y^I and the real variables x^I and y_I is complicated. Therefore we rewrite (66) in terms of the original variables Y^I and \bar{Y}^I , where explicit results are known,

$$\sum_{\{p, q\}} d(p, q) e^{\pi[q_I (Y + \bar{Y})^I - p^I (\hat{F} + \hat{\bar{F}})_I]} \sim \sum_{\text{shifts}} e^{\pi \mathcal{F}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})}. \quad (67)$$

Here \mathcal{F} equals the free energy (45) suitably modified with possible non-holomorphic corrections. The latter requires that F_I is changed into $\hat{F}_I = F_I + 2i\Omega_I$, as was demonstrated in [18]. It is important to note that both sides of (67) (as well as of (66)) are manifestly consistent with duality.

Again, it is possible to formally invert (67) by means of an inverse Laplace transform,

$$\begin{aligned} d(p, q) &\propto \int d(Y + \bar{Y})^I d(\hat{F} + \hat{\bar{F}})_I e^{\pi \Sigma(Y, \bar{Y}, p, q)} \\ &\propto \int dY d\bar{Y} \Delta^-(Y, \bar{Y}) e^{\pi \Sigma(Y, \bar{Y}, p, q)}, \end{aligned} \quad (68)$$

where $\Delta^-(Y, \bar{Y})$ is an integration measure whose form depends on $\hat{F}_I + \hat{\bar{F}}_I$. The expression for Δ^- , as well as for a related determinant Δ^+ , reads as follows,

$$\Delta^\pm(Y, \bar{Y}) = \left| \det \left[\text{Im } F_{KL} + 2 \text{Re}(\Omega_{KL} \pm \Omega_{\bar{K}\bar{L}}) \right] \right|. \quad (69)$$

As before, F_{IJ} and F_I refer to Y -derivatives of the holomorphic function $F(Y, \Upsilon)$ whereas Ω_{IJ} and $\Omega_{I\bar{J}}$ denote the holomorphic and mixed holomorphic-antiholomorphic second derivatives

⁴In case that the Hesse potential exhibits a periodicity with a different periodicity interval, then the sum over the imaginary shifts will have to be modded out appropriately such as to avoid overcounting.

of Ω , respectively. In the absence of non-holomorphic corrections $\Delta^+ = \Delta^-$.

A priori it is not clear whether the integral (68) is well-defined and we refer to [18] for a discussion. Note that the periodicity sum in (68) is defined in terms of the variables ϕ and χ , which should have some bearing on the integration contour in (68). Leaving aside these subtle points one may consider a saddle-point approximation of the intergral representation (68). In view of the previous results it is clear that the saddle point coincides with the attractor point. Subsequently one evaluates the semiclassical Gaussian integral that emerges when expanding the exponent in the integrand to second order in δY^I and $\delta \bar{Y}^I$ about the attractor point. When $Y^I - \bar{Y}^I = ip^I$, the resulting determinant factorizes into the square roots of two subdeterminants, $\sqrt{\Delta^+}$ and $\sqrt{\Delta^-}$. Here the plus (minus) sign refers to the contribution of integrating over the real (imaginary) part of δY^I . Consequently, the result of a saddle-point approximation applied to (68) yields,

$$d(p, q) = \sqrt{\left| \frac{\Delta^-(Y, \bar{Y})}{\Delta^+(Y, \bar{Y})} \right|_{\text{attractor}}} e^{\mathcal{S}_{\text{macro}}(p, q)} . \quad (70)$$

In the absence of non-holomorphic corrections the ratio of the two determinants is equal to unity and one recovers precisely the macroscopic entropy. In the presence of non-holomorphic terms, the deviations from unity are usually suppressed in the limit of large charges, and one recovers the leading and subleading corrections to the entropy [18]. Of course, this is only the case when the saddle-point approximation is appropriate.

Alternatively one may choose to integrate only over the imaginary values of the fluctuations δY^I in saddle-point approximation. The saddle point then occurs in the subspace defined by the magnetic attractor equations, so that one obtains a modified version of the OSV integral [36],

$$d(p, q) \propto \int d\phi \sqrt{\Delta^-(p, \phi)} e^{\pi[\mathcal{F}_E(p, \phi) - q_I \phi^I]} , \quad (71)$$

where $\mathcal{F}_E(p, \phi)$ was defined in (55) and $\Delta^-(p, \phi)$ is defined in (69) with the Y^I given by (53).

Hence this integral contains a non-trivial integration measure factor $\sqrt{\Delta^-}$ in order to remain consistent with electric/magnetic duality. Without the integral measure this is the integral conjectured in [36]. In view of the original setting in terms of the Hesse potential, we expect that the integration contours in (71) should be taken along the imaginary axes.

Inverting (71) to a partition sum over a mixed ensemble, one finds,

$$\begin{aligned} Z(p, \phi) &= \sum_{\{q\}} d(p, q) e^{\pi q_I \phi^I} \\ &\sim \sum_{\text{shifts}} \sqrt{\Delta^-(p, \phi)} e^{\pi \mathcal{F}_E(p, \phi)} . \end{aligned} \quad (72)$$

It should be noted that this expression and the preceding one is less general than (68) because it involves a saddle-point approximation. Moreover the function \mathcal{F}_E is not duality invariant and the invariance is only recaptured when completing the saddle-point approximation with respect to the fields ϕ^I . Therefore an evaluation of (71) beyond the saddle-point approximation will most likely give rise to a violation of (some of) the duality symmetries again.

7.1. The integration measure and the mixed partition function

As mentioned earlier, the partition function $Z(p, \phi)$ was originally conjectured to be equal to the modulus square of the partition function of the topological string [36]. Soon thereafter, however, it was realized that this relationship must be more subtle. The arguments in [18], based on electric/magnetic duality clearly indicate some of the missing ingredients, resulting in the measure factor $\sqrt{\Delta^-}$ and in the presence of the non-holomorphic corrections. In fact, it was already clear from an early analysis of small heterotic black holes that T-duality was not conserved when straightforwardly applying these ideas [52]. Although the presence of the measure factor corrects for the lack of duality invariance, the semiclassical results for small black holes seem to remain inconsistent with the analysis of microstate counting, a fact we already alluded to at the end of subsection 6.2.

It is possible to test the result (72) in the context of the 1/4-BPS states of the heterotic $N = 4$ supersymmetric string compactifications, by making use of the degeneracy formula (61). Such a calculation was first performed in [53] and it was subsequently generalized in [18] for more general charge configurations and for CHL models. Using the degeneracies (61) one calculates the mixed partition sum on the left-hand side of (72). As it turns out, the resulting expression is indeed proportional to the square of the partition function of the topological string,

$$|Z_{\text{top}}(p, \phi)|^2 = e^{-2\pi i[F(Y, \Upsilon) - \bar{F}(\bar{Y}, \bar{\Upsilon})]}, \quad (73)$$

where $F(Y, \Upsilon)$ is now the *holomorphic* part of (56) and (59),

$$F(Y, \Upsilon) = -\frac{Y^1 Y^a \eta_{ab} Y^b}{Y^0} + \frac{i \Upsilon}{128\pi} \log \eta^{24}(-iY^1/Y^0). \quad (74)$$

Here $\Upsilon = -64$ and the Y^I are given by (53).⁵ However, there is a non-trivial proportionality factor, which, up to subleading contributions, we expect to coincide with

$$\sqrt{\Delta^-(p, \phi)} \exp[4\pi \Omega_{\text{nonholo}}]. \quad (75)$$

The expression for Ω_{nonholo} follows from the last term in (59), and is thus equal to

$$\exp[4\pi \Omega_{\text{nonholo}}] = (S + \bar{S})^{-12}, \quad (76)$$

where

$$S + \bar{S} = \frac{2(p^1 \phi^0 - p^0 \phi^1)}{(\phi^0)^2 + (p^0)^2}. \quad (77)$$

The factor $(S + \bar{S})^{-12}$ cancels against a similar factor in $\sqrt{\Delta^-}$ and, up to subleading terms, (75) becomes

$$\sqrt{\Delta^-(p, \phi)} \exp[4\pi \Omega_{\text{nonholo}}] \approx \frac{i(\bar{Y}^I \hat{F}_I - Y^I \hat{F}_I)}{2|Y^0|^2} = \frac{e^{-\mathcal{K}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})}}{2|Y^0|^2}. \quad (78)$$

Indeed this result coincides with the result found in [53, 18]. The expression for $e^{-\mathcal{K}}$ was already defined in (38) up to non-holomorphic corrections.

⁵For convenience, we only refer to the case $k = 10$.

The latter can be dropped as they are subleading. Note that $e^{-\mathcal{K}}$ has been rescaled by replacing the X^I and A by Y^I and Υ , respectively. This is to be expected in view of the fact that the right-hand side of (78) must be invariant under such rescalings.

However, we should discuss a subtlety related to the fact that we derived the expression for $\sqrt{\Delta^-}$ in the context of $N = 2$ supergravity, whereas the evaluation based on (61) is based on $N = 4$ supersymmetric compactifications. This means that the number of scalar moduli (related to the number of $N = 2$ vector multiplets) is not obviously the same. In the case of $N = 4$ the rank of the gauge group is equal to 28 (for simplicity we restrict ourselves to the case $k = 10$). Of the 28 abelian vector gauge fields, 6 will correspond to the graviphotons of pure $N = 4$ supergravity, and 22 will each belong to an $N = 4$ vector supermultiplet. In the reduction to $N = 2$ supergravity, one of the graviphotons will be contained in an additional $N = 2$ vector supermultiplet and another one will play the role of the graviphoton of $N = 2$ supergravity. There are thus 24 abelian vector gauge fields, of which 23 are associated with $N = 2$ vector multiplets and one is associated with the $N = 2$ graviphoton. The remaining 4 graviphotons are associated with two $N = 2$ gravitino supermultiplets, and these vector fields seem to have no place in the $N = 2$ description. Therefore it is often assumed that the effective rank of the gauge group in the $N = 2$ description should be taken equal to 24, rather than to 28.

Nevertheless, in the calculation of $\sqrt{\Delta^-}$ leading to (78) we assumed that the $N = 2$ description is based on 28 vector fields, corresponding to 27 vector supermultiplets and one graviphoton field. Only in that case, the factor $(S + \bar{S})^{-12}$ cancels so that one obtains the proportionality constant noted in (78) based on the $N = 2$ expression for $\sqrt{\Delta^-}$. This somewhat confusing issue seems entirely due to the fact that the $N = 2$ description of an $N = 4$ theory is not fully understood, as the corresponding description of $N = 4$ supergravity is indeed based on 28 electrostatic potentials ϕ which are contained in 28, rather than 24, analogues of the $N = 2$ quantities Y^I .

In a recent series of papers [54,55,56] progress was made towards a further understanding of the relation between the mixed black hole partition function and the partition function of the topological string. Unfortunately no evidence for the presence of the integration measure factor in (71) and (72) was presented. However, a more detailed analysis for compact Calabi-Yau [57] models subsequently revealed the presence of the measure factor. Based on an extensive analysis of the factorization formula for BPS indices, it is shown that the partition function does not completely factorize into a holomorphic and an anti-holomorphic sector and the measure that is found agrees (for $p^0 = 0$) with (78). The power of $|Y^0|$ depends on whether one is discussing $N = 2$ or $N = 4$ black holes. These results seem to be in line with what was discussed in this section, although there are many subtleties. Obviously more work is needed to fully explore their consequences.

The work described in these lectures is based on various collaborations with Gabriel Lopes Cardoso, Jürg Käppeli, Swapna Mahapatra, Thomas Mohaupt and Frank Saueressig. I thank Gabriel Lopes Cardoso for discussion and valuable comments on the manuscript.

This work is partly supported by NWO grant 047017015, EU contracts MRTN-CT-2004-005104 and MRTN-CT-2004-512194, and INTAS contract 03-51-6346.

REFERENCES

1. S.W. Hawking, *Comm. Math. Phys.* **43** (1975) 199.
2. J.M. Bardeen, B. Carter and S.W. Hawking, *Commun. Math. Phys.* **31** (1973) 161.
3. J.D. Bekenstein, *Phys. Rev.* **D7** (1973) 2333; *Phys. Rev.* **D9** (1974) 3292.
4. B. Pioline, *Class. Quant. Grav.* **23** (2006) S981, [hep-th/0607227](#).
5. F. Larsen, [hep-th/0608191](#).
6. P. Kraus, [hep-th/0609074](#).
7. T. Mohaupt, [hep-th/0703035](#).
8. A. Strominger and C. Vafa, *Phys. Lett.* **B379** (1996) 99, [hep-th/9601029](#).
9. J.M. Maldacena, A. Strominger and E. Witten, *JHEP* **12** (1997) 002, [hep-th/9711053](#).
10. C. Vafa, *Adv. Theor. Math. Phys.* **2** (1998) 207, [hep-th/9711067](#).
11. R. Minasian, G. Moore and D. Tsimpis, *Commun. Math. Phys.* **209** (2000) 325, [hep-th/0009042](#).
12. S. Ferrara, R. Kallosh and A. Strominger, *Phys. Rev.* **D52** (1995) 5412, [hep-th/9508072](#).
13. A. Strominger, *Phys. Lett.* **B383** (1996) 39, [hep-th/9602111](#).
14. S. Ferrara and R. Kallosh, *Phys. Rev.* **D54** (1996) 1514, [hep-th/9602136](#).
15. G.L. Cardoso, B. de Wit, J. Käppeli and T. Mohaupt, *JHEP* **12** (2000) 019, [hep-th/0009234](#).
16. R.M. Wald, *Phys. Rev.* **D48** (1993) 3427, [gr-qc/9307038](#); V. Iyer and R.M. Wald, *Phys. Rev.* **D50** (1994) 846, [gr-qc/9403028](#); T. Jacobson, G. Kang and R.C. Myers, *Phys. Rev.* **D49** (1994) 6587, [gr-qc/9312023](#).
17. K. Behrndt, G.L. Cardoso, B. de Wit, R. Kallosh, D. Lüst and T. Mohaupt, *Nucl. Phys.* **B488** (1997) 236, [hep-th/9610105](#).
18. G.L. Cardoso, B. de Wit, J. Käppeli and T. Mohaupt, *JHEP* **03** (2006) 074, [hep-th/0601108](#).
19. A. Sen, *JHEP* **0509** (2005) 038, [hep-th/0506177](#).
20. S. Ferrara, G.W. Gibbons and R. Kallosh, *Nucl. Phys.* **B500** (1997) 75-93, [hep-th/9702103](#).
21. G.W. Gibbons, in: *Duality and Supersymmetric Theories*, eds. D.I. Olive and P.C. West, Cambridge (1997) 267.
22. K. Goldstein, N. Iizuka, R.P. Jena and S.P. Trivedi, *Phys. Rev.* **D72** (2005) 124021, [hep-th/0507096](#).
23. M.K. Gaillard and B. Zumino, *Nucl. Phys.* **B193** (1981) 221.
24. B. de Wit, *Nucl. Phys. Proc. Suppl.* **101** (2001) 154, [hep-th/0103086](#).
25. G.L. Cardoso, B. de Wit and S. Mahapatra, *JHEP* **03** (2007) 085, [hep-th/0612225](#).
26. B. de Wit and A. Van Proeyen, *Nucl. Phys.* **B245** (1984) 89; B. de Wit, J.-W. van Holten and A. Van Proeyen, *Nucl. Phys.* **B184**

- (1981) 77; B. de Wit, P.G. Lauwers and A. Van Proeyen, Nucl. Phys. **B255** (1985) 569.
27. E.A. Bergshoeff, M. de Roo and B. de Wit, Nucl. Phys. **B182** (1981) 173.
28. B. de Wit, Nucl. Phys. Proc. Suppl. **49** (1996) 191-200, [hep-th/9602060](#); Fortschr. Phys. **44** (1996) 529-538, [hep-th/9603191](#).
29. G.L. Cardoso, B. de Wit and T. Mohaupt, Phys. Lett. **B451** (1999) 309, [hep-th/9812082](#).
30. G.L. Cardoso, B. de Wit and T. Mohaupt, Fortsch. Phys. **48** (2000) 49, [hep-th/9904005](#).
31. G.L. Cardoso, B. de Wit and T. Mohaupt, Nucl. Phys. **B567** (2000) 87, [hep-th/9906094](#).
32. B. de Wit and F. Saueressig, JHEP **0609** (2006) 062, [hep-th/0606148](#).
33. G. Exirifard, JHEP **10** (2006) 070, [hep-th/0607094](#).
34. B. Sahoo and A. Sen, [hep-th/0608182](#).
35. G.L. Cardoso, B. de Wit, J. Käppeli and T. Mohaupt, JHEP **12** (2004) 075, [hep-th/0412287](#).
36. H. Ooguri, A. Strominger and C. Vafa, “Black hole attractors and the topological string”, *Phys. Rev.* **D70** (2004) 106007, [hep-th/0405146](#).
37. M. Berhadsky, S. Cecotti, H. Ooguri and C. Vafa, “Kodeira-Spencer theory of gravity and exact results for quantum string amplitudes”, *Commun. Math. Phys.* **165** (1994) 311, [hep-th/9309140](#).
38. J.A. Harvey and G.W. Moore, *Phys. Rev.* **D57** (1998) 2323 [hep-th/9610237](#).
39. M. Cvetič and D. Youm, *Phys. Rev.* **D53** (1996) 584 ([hep-th/9507090](#)).
40. M. Cvetič and A.A. Tseytlin, *Phys. Rev.* **D53** (1996) 5619 ([hep-th/9512031](#)).
41. E.A. Bergshoeff, R. Kallosh and T. Ortin, Nucl. Phys. **B478** (1996) 156, [hep-th/9605059](#).
42. S. Chaudhuri, G. Hockney and J.D. Lykken, *Phys. Rev. Lett.* **75** (1995) 2264, [hep-th/9505054](#).
43. R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. **B484** (1997) 543, [hep-th/9607026](#).
44. D.P. Jatkar and A. Sen, [hep-th/0510147](#).
45. A. Dabholkar and J.A. Harvey, *Phys. Rev. Lett.* **63** (1989) 478.
46. J.G. Russo and L. Susskind, Nucl. Phys. **B437** (1995) 611, [hep-th/9405117](#).
47. A. Sen, Nucl. Phys. **B440** (1995) 421, [hep-th/9411187](#).
48. A. Sen, *Mod. Phys. Lett.* **A10** (1995) 2081, [hep-th/9504147](#).
49. A. Peet, Nucl. Phys. **B456** (1995) 732, [hep-th/9506200](#).
50. A. Dabholkar, [hep-th/0409148](#).
51. A. Dabholkar, R. Kallosh and A. Maloney, JHEP **0412** (2004) 059, [hep-th/0410076](#).
52. A. Dabholkar, F. Denef, G.W. Moore and B. Pioline, JHEP **0510** (2005) 096, [hep-th/0507014](#).
53. D. Shih and X. Yin, JHEP **0604** (2006) 034 [hep-th/0508174](#).
54. D. Gaiotto, A. Strominger and X. Yin, [hep-th/0602046](#).
55. C. Beasley, D. Gaiotto, M. Guica, L. Huang, A. Strominger and X. Yin, [hep-th/0608021](#).
56. J. de Boer, M. Cheng, R. Dijkgraaf, J. Manschot and E. Verlinde, JHEP **0611** (2006) 024 [hep-th/0608059](#).
57. F. Denef and G. Moore, [hep-th/0702146](#).